LOGARITHMIC BLOCH SPACE AND ITS PREDUAL

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ABSTRACT. We consider the space $\mathfrak{B}^1_{\log^\alpha}$, of analytic functions on the unit disk $\mathbb{D}$, defined by the requirement $\int_{\mathbb{D}} |f'(z)|\phi(|z|)\, dA(z) < \infty$, where $\phi(r) = \log^\alpha(1/(1-r))$ and show that it is a predual of the "$\log^\alpha$-Bloch" space and the dual of the corresponding little Bloch space. We prove that a function $f(z) = \sum_{n=0}^\infty a_n z^n$ with $a_n \downarrow 0$ is in $\mathfrak{B}^1_{\log^\alpha}$ iff $\sum_{n=0}^\infty \log^\alpha(n+2)/(n+1) < \infty$ and apply this to obtain a criterion for membership of the Libera transform of a function with positive coefficients in $\mathfrak{B}^1_{\log^\alpha}$. Some properties of the Cesàro and the Libera operator are considered as well.

1. INTRODUCTION AND SOME RESULTS

Let $H(\mathbb{D})$ denote the space of all functions analytic in the unit disk $\mathbb{D}$ of the complex plane. Endowed with the topology of uniform convergence on compact subsets of $\mathbb{D}$, the class $H(\mathbb{D})$ becomes a complete locally convex space. In this paper we are concerned with the predual of the space $\mathfrak{B}_{\log^\alpha}$, $\alpha \in \mathbb{R}$,

$$(1.1) \qquad \mathfrak{B}_{\log^\alpha} = \left\{ f \in H(\mathbb{D}) : |f'(z)| = \mathcal{O}\Big((1-|z|)^{-1} \log^\alpha \frac{2}{1-|z|} \Big) \right\}.$$

The norm in $\mathfrak{B}_{\log^\alpha}$ is defined by

$$\|f\|_{\mathfrak{B}_{\log^\alpha}} = |f(0)| + \sup_{z \in \mathbb{D}} (1-|z|) \log^{-\alpha} \frac{1}{1-|z|}.$$

The subspace, $\mathfrak{b}_{\log^\alpha}$, of $\mathfrak{B}_{\log^\alpha}$ is defined by replacing "$\mathcal{O}$" with "o". It will be proved:

(A) The dual of $\mathfrak{b}_{\log^\alpha}$ is isomorphic to $\mathfrak{B}^1_{\log^\alpha}$,

$$(1.2) \qquad \mathfrak{B}^1_{\log^\alpha} = \left\{ f : \|f\|_{\mathfrak{B}^1_{\log^\alpha}} = |f(0)| + \int_{\mathbb{D}} |f'(z)| \log^\alpha \frac{2}{1-|z|}\, dA(z) < \infty \right\},$$

and the dual of $\mathfrak{B}^1_{\log^\alpha}$ is isomorphic to $\mathfrak{B}_{\log^\alpha}$, in both cases with respect to the bilinear form

$$(1.3) \qquad \langle f, g \rangle = \lim_{r \uparrow 1} \sum_{n=0}^\infty \hat{f}(n) \hat{g}(n) r^{2n}.$$

(In (1.2) dA stands for the normalized Lebesgue measure on $\mathbb{D}$.) This extends the well-known result on the Bloch space and the little Bloch space $\mathfrak{b} := \mathfrak{b}_{\log^0}$.

These spaces are Banach spaces, and the space $\mathfrak{b}_{\log^\alpha}$ coincides with the closure in $\mathfrak{B}_{\log^\alpha}$ of the set of all polynomials. The space $\mathfrak{B}_{\log} := \mathfrak{B}_{\log^1}$ occurs naturally in the study of pointwise multipliers on the usual Bloch space $\mathfrak{B} := \mathfrak{B}_{\log^0}$ (see [3]).

One of interesting properties of $\mathfrak{B}^1_{\log^\alpha}$ is described in the following theorem:

Theorem 1.1. *Let $f(z) = \sum_{n=0}^\infty a_n z^n$, where $\{a_n\}$ is a nonincreasing sequence, of real numbers, tending to zero. Let $\alpha \geq -1$. Then f belongs to $\mathfrak{B}^1_{\log^\alpha}$ if and only if*

$$(1.4) \qquad S_\alpha(f) := \sum_{n=0}^\infty \frac{a_n \log^\alpha(n+2)}{n+1} < \infty.$$

Moreover, there is a constant C independent of $\{a_n\}$ such that $S_\alpha(f)/C \leq \|f\|_{\mathfrak{B}^1_{\log^\alpha}} \leq CS_\alpha(f)$.

Proof. See Section 4. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

In the case $\alpha = 0$, this assertion is proved in [17]. We can take a_n to be the coefficients of the Libera transform of a function with positive coefficients. Namely, if $g(z) = \sum_{n=0}^\infty \hat{g}(n) z^n$ and

$$(1.5) \qquad\qquad \sum_{n=0}^\infty \frac{|\hat{g}(n)|}{n+1} < \infty,$$

then the Libera transform $\mathcal{L}g$ of g is well defined as

$$(1.6) \qquad \mathcal{L}g(z) = \frac{1}{1-z} \int_z^1 f(\zeta)\, d\zeta$$
$$= \sum_{n=0}^\infty z^n \sum_{k=n}^\infty \frac{\hat{g}(k)}{k+1}$$

(see, e.g., [12]). If $\hat{g} \geq 0$, then condition (1.5) is also necessary for the existence of the integral in (1.10): take $z = 0$ to conclude that (1.10) implies the convergence of the integral

$$\int_0^1 g(t)\, dt = \sum_{n=0}^\infty \frac{\hat{g}(n)}{n+1}.$$

Then, as an application of Theorem 1.1 we get:

Theorem 1.2. *Let $\alpha > -1$, let $g \in H(\mathbb{D})$, and $\hat{g} \geq 0$. Then $\mathcal{L}g$ is in $\mathfrak{B}^1_{\log^\alpha}$ if and only if*

$$(1.7) \qquad K_\alpha(g) := \sum_{n=0}^\infty \frac{\hat{g}(n) \log^{\alpha+1}(n+2)}{n+1} < \infty.$$

We have $K_\alpha(g)/C \leq \|\mathcal{L}g\|_{\mathfrak{B}^1_{\log^\alpha}} \leq CK_\alpha(g)$.

Proof. See Section 4. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

In the general case, the integral in (1.6) need not exists, but it certainly exists if $g \in H(\overline{\mathbb{D}})$, which means that g is analytic in a neighborhood of the closed disk. By using Theorem 1.1 we shall prove that $\overline{\mathcal{L}}$ cannot be extended to a bounded operator from $\mathfrak{B}^1_{\log^\alpha}$ to $H(\mathbb{D})$, if $\alpha < 0$. In the case $\alpha \geq 0$, every function $g \in \mathfrak{B}^1_{\log^\alpha}$ satisfies (1.5), whence $\mathcal{L}$ is well defined, and we will show that $\mathcal{L}$ maps this space into $\mathfrak{B}^1_{\log^{\alpha-1}}$, when $\alpha > 0$. If $\alpha = 0$ we need a sort of "iterated" logarithmic space.

Cesàro operator. The dual of $H(\mathbb{D})$ is equal to $H(\overline{\mathbb{D}})$, where "$g \in H(\overline{\mathbb{D}})$" means that g is holomorphic in a neighborhood of $\overline{\mathbb{D}}$ (depending on g). The duality pairing is given

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \hat{f}(n)\hat{g}(n), \tag{1.8}$$

where $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n \in H(\mathbb{D})$ and $g(z) = \sum_{n=0}^{\infty} \hat{g}(n)z^n \in H(\overline{\mathbb{D}})$, and the series is absolutely convergent (see, e.g., [8]). The Cesàro operator is defined on $H(\mathbb{D})$ as

$$\mathcal{C}f(z) = \sum_{n=0}^{\infty} z^n \frac{1}{n+1} \sum_{k=0}^{n} a_k, \quad f \in H(\mathbb{D}). \tag{1.9}$$

It is easy to verify that the adjoint of $\mathcal{C} : H(\mathbb{D}) \mapsto H(\mathbb{D})$ is equal to $\overline{\mathcal{L}} : H(\overline{\mathbb{D}}) \mapsto H(\overline{\mathbb{D}})$, under the pairing (1.8), and vice versa (see, e.g., [12]).

The operators $\mathcal{C}$ and $\overline{\mathcal{L}}$ acting on H^p spaces were first studied by Siskakis in 1987. In [21] he proved that $\mathcal{C}$ is bounded on H^p for $1 < p < \infty$, and that $\overline{\mathcal{L}}$ can be extended to a bounded operator on H^p, $1 < p < \infty$, and obtained some results on their spectra and norms. A few years later he proved the boundedness of the Cesàro operator on H^1 ([22]), while Miao proved its boundedness on H^p for $0 < p < 1$ ([10]). A short proof of the boundedness of $\mathcal{C}$ on H^p, $0 < p < \infty$, as well as a stronger result, can be also found in Nowak [11]. However, H^∞ is not mapped into itself by $\mathcal{C}$ (see [4]). If we write (1.9) as

$$z\mathcal{C}f(z) = \int_0^z \frac{f(\zeta)}{1-\zeta}\,d\zeta,$$

and hence

$$(z\mathcal{C}f(z))' = \frac{f(z)}{1-z},$$

we conclude that $\mathcal{C}$ maps H^∞ into the Bloch space (see [4]).

On the other hand, by using the inequality

$$|f(z)| = \mathcal{O}\Big(\log \frac{2}{1-|z|} \Big), \quad f \in \mathfrak{B},$$

and the analogous inequality for $f \in \mathfrak{b}$ (replace "$\mathcal{O}$" with "o"), we get:

(C) The operator $\mathcal{C}$ maps the space $\mathfrak{B}$ into $\mathfrak{B}_{\log}$, and $\mathfrak{b}$ into $\mathfrak{b}_{\log}$.

One of our aims is to generalize this assertion to some other values of α and then use assertion **(A)** together with the duality between $\mathcal{C}$ and $\overline{\mathcal{L}}$ to obtain an alternative proof of some results on the action of $\mathcal{L}$ from $\mathfrak{B}^1_{\alpha+1}$ to $\mathfrak{B}^1_\alpha$, where

$$\mathcal{L}f(z) = \int_0^1 f(t + (1-t)z)\,dt. \tag{1.10}$$

In particular we have:

(D) The operator $\mathcal{L}$ is well defined on $\mathfrak{B}^1_{\log}$ and maps it into $\mathfrak{B}^1$.

It should be noted that: (a) $\mathfrak{B}^1 \subsetneq H^1$; (b) $\mathcal{L}$ does not map $\mathfrak{B}^1$ into H^1 (see [17]); and (c) $\mathcal{L}$ maps $\mathfrak{B}$ into BMOA [12], which improves an earlier result, namely that $\mathcal{L}$ maps $\mathfrak{B}$ into $\mathfrak{B}$ ([5, 24]).

The formula (1.10) is obtained from (1.6) by integrating over the straight line joining z and 1. A sufficient (not necessary [18]) condition for the possibility of such integration is (1.5) ($g = f$).

In proving some of our results, in particular assertions **(A)** and **(B)**, we use a sequence of polynomials constructed in [6] (see also [7] and [16]) to decompose the space into a sum which resembles a sum of finite-dimensional spaces (see Section 3).

2. SOME MORE RESULTS

Some elementary facts concerning the cases when $\mathcal{L}f$ is well defined are collected in the following theorem, where

$$\ell^1_{-1} = \Big\{ g \in H(\mathbb{D}) : \|g\|_{\ell^1_{-1}} = \sum_{n=0}^{\infty} \frac{|\hat{g}(n)|}{n+1} < \infty \Big\}.$$

Theorem 2.1. *Let $\alpha \in \mathbb{R}$. Then:*

(a) $\mathfrak{B}_{\log^\alpha} \subset \ell^1_{-1}$ *for all α;*

(b) $\mathfrak{B}^1_{\log^\alpha} \subset \ell^1_{-1}$ *if and only if $\alpha \geq 0$;*

(c) *if $\alpha < 0$, then $\overline{\mathcal{L}}$ cannot be extended to a continuous operator from $\mathfrak{B}^1_{\log^\alpha}$ to $H(\mathbb{D})$.*

Proof. See Section 4.   □

Remark 2.1. The inclusions in (a) and (b) are continuous. Assertion (c) says much more than simply that $\mathfrak{B}^1_{\log^\alpha} \not\subset \ell^1_{-1}$.

In the context of the action of $\mathcal{C}$ and $\mathcal{L}$ some new spaces occur: the space $\mathfrak{B}_{\log g}$ is defined by the requirement

$$|f'(z)| = \mathcal{O}\Big(\log \log \frac{4}{1-|z|} \Big),$$

the space $\mathfrak{b}_{\log g}$ defined by replacing "$\mathcal{O}$" with "o", and the space $\mathfrak{B}^1_{\log g}$ defined by

$$\int_{\mathbb{D}} |f'(z)| \log \log \frac{4}{1-|z|} \, dA(z) \; < \infty.$$

Our next result is

Theorem 2.2. (a) *If $\alpha > -1$, then $\mathcal{C}$ maps the space $\mathfrak{B}_{\log^\alpha}$, resp. $\mathfrak{b}_{\log^\alpha}$, into $\mathfrak{B}_{\log^{\alpha+1}}$, resp. $\mathfrak{b}_{\log^{\alpha+1}}$.*

(b) *$\mathcal{C}$ maps the space $\mathfrak{B}_{\log^{-1}}$, resp. $\mathfrak{b}_{\log^{-1}}$, into $\mathfrak{B}_{\log g}$, resp. $\mathfrak{b}_{\log g}$.*

Proof. See Section 5.   □

Remark 2.2. If $f \in \mathfrak{B}_{\log^\alpha}$ and $\alpha < -1$, then, as it can easily be shown, $f \in A(\mathbb{D})$, where $A(\mathbb{D})$ is the disk-algebra, i.e., the subset of H^∞ consisting of those f which have a continuous extension to the closed disk. Moreover, the modulus of continuity of the boundary function $f_*(\zeta)$, $\zeta \in \partial\mathbb{D}$, satisfies the condition

$$\omega(f_*, t) = \mathcal{O}\Big(t \Big(\log \frac{2}{t} \Big)^{\alpha+1} \Big), \quad t \downarrow 0.$$

This follows from the inequality

$$\omega(f_*, t) \leq C \int_{1-t}^{1} M_\infty(r, f') \, dr,$$

see [15, Theorem 2.2]. It should be noted that the modulus of continuity of f_* is "proportional" to that of $f(z)$, $z \in \mathbb{D}$, see [23, 19].

Concerning the Libera operator we shall prove, besides Theorem 2.1(c), the following facts.

Theorem 2.3. (a) *If $\alpha > 0$, then $\mathcal{L}$ is well defined on $\mathfrak{B}^1_{\log^\alpha}$ and maps this space to $\mathfrak{B}^1_{\log^{\alpha-1}}$.*

(b) *$\mathcal{L}$ is well defined on $\mathfrak{B}^1_{\log g}$ and maps this space into $\mathfrak{B}^1_{\log^{-1}}$.*

(c) *$\mathcal{L}$ is well defined on $\mathfrak{B}^1$ and maps it into $\mathfrak{B}^1_\alpha$ for all $\alpha < -1$.*

Proof. See Section 5. $\qquad\square$

Theorem 2.4. *Let $\alpha \in \mathbb{R}$. Then the dual of $\mathfrak{b}_{\log^\alpha}$, resp. $\mathfrak{B}^1_{\log^\alpha}$, is isomorphic to $\mathfrak{B}^1_{\log^\alpha}$, resp. $\mathfrak{B}_{\log^\alpha}$ under the pairing (1.3). Similarly, the dual of $\mathfrak{b}_{\log g}$, resp. $\mathfrak{B}^1_{\log g}$, is isomorphic to $\mathfrak{B}^1_{\log g}$, resp. $\mathfrak{B}_{\log g}$, under the same pairing.*

Proof. See Section 6. $\qquad\square$

Remark 2.3. The phrase "the dual of X is isomorphic to Y under the pairing (1.3)" means that if $f \in X$ and $g \in Y$, then the limit in (1.3) exists and the functional $\Phi(f) = \langle f, g \rangle$ is bounded on X; and on the other hand, if $\Phi \in X^*$, then there exists $g \in Y$ such that $\Phi(f) = \langle f, g \rangle$, and moreover, there exists a constant C independent of g such that $\|g\|_Y / C \leq \|\Phi\| \leq C\|g\|_Y$.

As an application of Theorems 2.2, 2.3, and 2.4, one can prove the following fact.

Theorem 2.5. *Let $\alpha > 0$. Then the adjoint (with respect to (1.3)) of the operator $\mathcal{L} : \mathfrak{B}^1_{\log^\alpha} \mapsto \mathfrak{B}^1_{\log^{\alpha-1}}$ is equal to $\mathcal{C} : \mathfrak{B}_{\log^{\alpha-1}} \mapsto \mathfrak{B}_{\log^\alpha}$. The adjoint of the operator $\mathcal{C} : \mathfrak{b}_{\log^{\alpha-1}} \mapsto \mathfrak{b}_{\log^\alpha}$ is equal to $\mathcal{L} : \mathfrak{B}^1_{\log^\alpha} \mapsto \mathfrak{B}^1_{\log^{\alpha-1}}$. The analogous assertions hold in the case when $\alpha = 0$.*

3. Decompositions

In [6], a sequence $\{V_n\}_0^\infty$ was constructed in the following way.
Let ω be a C^∞-function on $\mathbb{R}$ such that

(1) $\omega(t) = 1$ for $t \leq 1$,
(2) $\omega(t) = 0$ for $t \geq 2$,
(3) ω is decreasing and positive on the interval $(1, 2)$.

Let $\varphi(t) = \omega(t/2) - \omega(t)$, and let $V_0(z) = 1 + z$, and, for $n \geq 1$,

$$V_n(z) = \sum_{k=0}^{\infty} \varphi(k/2^{n-1})z^k = \sum_{k=2^{n-1}}^{2^{n+1}-1} \varphi(k/2^{n-1})z^k.$$

The polynomials V_n have the following properties:

(3.1) $$g(z) = \sum_{n=0}^{\infty} V_n * g(z), \quad \text{for } g \in H(\mathbb{D});$$

(3.2) $$\|V_n * g\|_p \leq C\|g\|_p, \quad \text{for } g \in H^p, \ p > 0;$$

(3.3) $$\|V_n\|_p \asymp 2^{n(1-1/p)}, \quad \text{for all } p > 0,$$

where $*$ denotes the Hadamard product. Here $\|h\|_p$ denotes the norm in the p-Hardy space H^p,

$$\|h\|_p = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\theta})| \, d\theta \right)^{1/p}$$
$$= \sup_{0<r<1} M_p(r,g).$$

We need additional properties.

Lemma 3.1. *Let* $P(z) = \sum_{k=m}^{j} a_k z^k$, $m < j$. *Then*

$$r^j \|P\|_p \leq M_p(r,P) \leq r^m \|P\|_p, \quad 0 < r < 1.$$

When applied to the polynomial $P = V_n * g'$, this gives:

$$(3.4) \qquad r^{2^{n+1}-1} \|V_n * g'\|_p \leq M_p(r, V_n * g') \leq r^{2^{n-1}-1} \|V_n * g'\|_p \text{ for } n \geq 1.$$

Another inequality will be used (see [16, Exercise 7.3.5]):

$$(3.5) \qquad 2^{n-1} \|V_n * g\|_p / C \leq \|V_n * g'\|_p \leq C 2^{n+1} \|V_n * g\|_p \text{ for } n \geq 1,$$

where C is a constant independent of n and g.

Theorem 3.1. *Let* $\alpha \in \mathbb{R}$, *and* $f \in H(\mathbb{D})$. *Then:*

(i) $f \in \mathfrak{B}_{\log^\alpha}$ *if and only if* $\sup_{n \geq 0} (n+1)^{-\alpha} \|V_n * f\|_\infty < \infty$.

(ii) $f \in \mathfrak{b}_{\log^\alpha}$ *if and only if* $\lim_{n \to \infty} (n+1)^{-\alpha} \|V_n * f\|_\infty = 0$.

(iii) $f \in \mathfrak{B}^1_{\log^\alpha}$ *if and only if* $\sum_{n=0}^{\infty} (n+1)^\alpha \|V_n * f\|_1 < \infty$.

Moreover, the inequality

$$C^{-1} \|f\|_{\mathfrak{B}_{\log^\alpha}} \leq \sup_{n \geq 0} (n+1)^{-\alpha} \|V_n * g\|_\infty \leq C^{-1} \|f\|_{\mathfrak{B}_{\log^\alpha}}$$

holds, where C *is independent of* f. *The analogous inequality holds in the case of* (iii) *as well.*

For the proof we need the following reformulation of [9, Proposition 4.1].

Lemma 3.2. *Let* φ *be a continuous function on the interval* $(0,1]$ *such that* $\varphi(x)/x^\gamma$ $(0 < x < 1)$ *is nonincreasing, and* $\varphi(x)/x^\beta$ $(0 < x < 1)$ *is nondecreasing, where* β *and* γ *are positive constants independent of* x. [*] *Let*

$$F_1(r) = (1-r)^{-1/q} \varphi(1-r) \sup_{n \geq 1} \lambda_n r^{2^{n+1}-1},$$

$$F_2(r) = (1-r)^{1/q} \varphi(1-r) \sum_{n=0}^{\infty} \lambda_n r^{2^{n-1}-1},$$

where $\lambda_n \geq 0$, $0 < q \leq \infty$. *If* $F = F_1$ *or* $F = F_2$, *then*

$$C^{-1} \|F\|_{L^q(0,1)} \leq \|\{\varphi(2^{-n}) \lambda_n\}\|_{\ell^q} \leq C \|F\|_{L^q(0,1)}.$$

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[*] *Following Shields and Williams* [20]*, we call such a function* **normal**.

Proof of Theorem 3.1. Case (i). Let $\varphi(x) = x \log^{-\alpha}(2/x)$, and $q = \infty$. That φ is normal follows from from the relation

$$\lim_{x \downarrow 0} \frac{x\varphi'_\alpha(x)}{\varphi_\alpha(x)} = 1.$$

Let $\lambda_n = 2^n \|V_n * f\|_\infty$. By (3.1), (3.2), (3.4), and (3.5), we have

$$C^{-1}|\hat{f}(1)| + C^{-1}\sup_{n \geq 1} \lambda_n r^{2^{n+1}-1} \leq M_\infty(r, f') \leq C|\hat{f}(1)| + C\sum_{n=1}^{\infty} \lambda_n r^{2^{n-1}-1}.$$

Hence, by Lemma 3.2, we obtain the desired result.

Case (ii). In this case we can proceed in two ways:

1° Modify the proof of Lemma 3.2 to get the inequalities

$$C^{-1}\|F\|_{C_0[0,1]} \leq \|\{\varphi(2^{-n})\lambda_n\}\|_{\mathfrak{c}_0} \leq C\|F\|_{C_0[0,1]},$$

where $C_0[0,1] = \{u \in C[0,1] : u(1) = 0\}$ and $\mathfrak{c}_0$ is the set of the sequences tending to zero.

2° Consider the spaces $\mathfrak{b}_{\log^\alpha} \subset \mathfrak{B}_{\log^\alpha}$ and $X = \{f : \|V_n * f\| = o((n+1)^\alpha)\}$, which is, by assertion (i) and its proof, a subspace of a space Y isomorphic to $\mathfrak{B}_{\log^\alpha}$. It is not hard to show that the polynomials are dense in both $\mathfrak{b}_{\log^\alpha}$ and X. This proves (ii).

Case (iii). In this case we use the function $\varphi(x) = x \log^\alpha(2/x)$, let $q = 1$, and then proceed as in the proof of (i). The details are omitted. This concludes the proof of the theorem. $\qquad\square$

Remark 3.1. By choosing $\phi(x) = x \log \log(\frac{4}{x})$, then we can conclude that Theorem 3.1 remains true if $\log^\alpha$, resp. $(n+1)^\alpha$, are replaced with $\log\log$, resp. $\log(n+2)$.

4. Functions with decreasing coefficients

Proof of Theorem 1.1. Assuming that (1.4) holds, we want to prove that

$$\|f\|_{\mathfrak{B}^1_{\log^\alpha}} \leq Ca_0 + C\sum_{n=1}^{\infty} a_{2^{n-1}}(n+1)^\alpha.$$

According to Theorem 3.1 and its proof, we have

$$C^{-1}\|f\|_{\mathfrak{B}^1_{\log^\alpha}} \leq a_0 + \sum_{n=1}^{\infty}(n+1)^\alpha \|V_n * f\|_1 \leq C\|f\|_{\mathfrak{B}^1_{\log^\alpha}}.$$

Let $n \geq 1$, $m = 2^{n-1}$, and $Q_k = \sum_{j=m}^{k} \varphi(j/m)e_j$. Since $Q_{4m-1} = V_n$, we have

$$V_n * f = \sum_{k=m}^{4m-1} \varphi(k/m)a_k e_k$$

$$= \sum_{k=m}^{4m-1} (a_k - a_{k+1})Q_k + a_{4m}Q_{4m-1}$$

$$= \sum_{k=m}^{4m-1} (a_k - a_{k+1})Q_k + a_{4m}V_n.$$

On the other hand, $Q_k = V_n * \Delta_{n,k}$, where

$$\Delta_{n,k} = \sum_{j=2^{n-1}}^{k} z^k, \quad 2^{n-1} \le k \le 2^{n+1}.$$

By (3.2), with $g = \Delta_{n,k}$, we have

$$\|Q_k\|_1 \le C\|\Delta_{n,k}\|_1 \le C\log(k+1-2^{n-1}) \le C(n+1).$$

Combining these inequalities we get

$$\|V_n * f\|_1(n+1)^\alpha \le C\sum_{k=m}^{4m-1}(a_k - a_{k+1})(n+1)^{\alpha+1} + Ca_{4m}\|V_n\|_1(n+1)^\alpha$$
$$\le C(n+1)^{\alpha+1}(a_m - a_{4m}) + Ca_{4m}(n+1)^\alpha$$
$$= C(n+1)^{\alpha+1}(a_{2^{n-1}} - a_{2^{n+1}}) + C(n+1)^\alpha a_{2^{n+1}}.$$

Here we have used the relation $\|V_n\|_1 \le C$ (see (3.3))! Thus

$$(n+1)^\alpha\|V_n * f\|_1 \le C(n+1)^{\alpha+1}(a_{2^{n-1}} - a_{2^n})$$
$$+ C(n+1)^{\alpha+1}(a_{2^n} - a_{2^{n+1}})$$
$$+ C(n+1)^\alpha a_{2^{n+1}},$$

and therefore it remains to estimate the sums

$$S_1 = \sum_{n=1}^{\infty}(n+1)^{\alpha+1}(a_{2^{n-1}} - a_{2^n}) \quad \text{and} \quad S_2 = \sum_{n=1}^{\infty}(n+1)^{\alpha+1}(a_{2^n} - a_{2^{n+1}}).$$

If $\alpha > -1$, then

$$(n+1)^{\alpha+1} \le C\sum_{k=1}^{n}(k+1)^\alpha,$$

and hence

$$S_1 \le C\sum_{n=1}^{\infty}(a_{2^{n-1}} - a_{2^n})\sum_{k=1}^{n}(k+1)^\alpha$$
$$= C\sum_{k=1}^{\infty}(k+1)^\alpha\sum_{n=k}^{\infty}(a_{2^{n-1}} - a_{2^n})$$
$$= C\sum_{k=1}^{\infty}(k+1)^\alpha_{2^{k-1}}.$$

In the case of S_2 we get

$$S_2 \le C\sum_{k=1}^{\infty}(k+1)^\alpha a_{2^k},$$

which completes the proof of "if" part of the theorem in the case $\alpha > -1$. If $\alpha = -1$, then

$$\|V_n * f\|_1(n+1)^{-1} \le C(a_{2^{n-1}} - a_{2^{n+1}}) + C(n+1)^{-1}a_{2^{n+1}},$$

from which we get the desired result in the case $\alpha = -1$.

To prove "only if" part we use Hardy's inequality in the form

$$\pi M_1(r,g) \ge \sum_{n=0}^{\infty}\frac{|\hat{g}(n)|}{n+1}r^n.$$

It follows that

$$\int_{\mathbb{D}} |f'(z)| \log^\alpha \frac{2}{1-|z|} \, dA(z)$$

$$= 2 \int_0^1 M_1(r, f') \log^\alpha \frac{2}{1-r} r \, dr$$

$$\geq \frac{2}{\pi} \sum_{n=1}^\infty a_n \frac{n}{n+1} \int_0^1 \log^\alpha \frac{2}{1-r} r^n \, dr.$$

Now the desired result follows from the inequality

$$\int_0^1 \frac{\varphi(1-r)}{1-r} r^n \, dr \geq c\varphi\left(\frac{1}{n+1}\right) \quad (c = \text{const.} > 0),$$

valid for any function normal function φ (see [9, Lemma 4.1]). □

Before proving Theorem 1.2, some remarks are in order. Let $f(z) = \sum_{n=0}^\infty a_n z^n$, $a_n \geq 0$. In order that $\mathcal{L}f$ be well defined by (1.10) *it is necessary and sufficient* that

(4.1) $$\sum_{n=0}^\infty \frac{a_n}{n+1} < \infty.$$

We already mentioned in Introduction that this condition implies the existence of the integral in (1.10). In fact, this integral converges uniformly on compact subsets of $\mathbb{D}$, which means that the limit

$$\lim_{x \uparrow 1} \int_0^x f(t + (1-t)z) \, dt$$

exists and is uniform in $|z| < \rho$, for every $\rho < 1$. This guarantees that $\mathcal{L}f$ is analytic. On the other hand, if the integral in (1.10) exists, then we take $z = 0$ to conclude that (4.1) holds.

Proof of Theorem 1.2. The Taylor coefficients of $\mathcal{L}f$ are

$$b_n = \sum_{k=n}^\infty \frac{a_n}{n+1}.$$

The sequence $\{b_n\}$ is nonincreasing so we can apply Theorem 1.1 to conclude that $\mathcal{L}f \in \mathfrak{B}^1_{\log^\alpha}$ if and only if

$$\sum_{n=0}^\infty \frac{\log^\alpha(n+2)}{n+1} \sum_{k=n}^\infty \frac{a_k}{k+1}$$

$$= \sum_{k=0}^\infty \frac{a_k}{k+1} \sum_{n=0}^k \frac{\log^\alpha(n+2)}{n+1} < \infty.$$

Now the desired result follows from the estimate

$$C^{-1} \log^{\alpha+1}(k+2) \leq \sum_{n=0}^k \frac{\log^\alpha(n+2)}{n+1} \leq C \log^{\alpha+1}(k+2),$$

which holds because $\alpha > -1$. □

Remark 4.1. The above proof shows that $\mathcal{L}f$ belongs to $\mathfrak{B}^1_{\log^{-1}}$ if and only if

$$\sum_{n=0}^{\infty} \frac{a_n}{\log\log(n+4)} < \infty.$$

Now we pass to the proof of Theorem 2.1.

Proof of Theorem 1.1(c). Since $\mathfrak{B}_{\log^\alpha} \subset \mathfrak{B}_{\log^\beta}$ for $\beta < \alpha$, we may assume that $-1 < \alpha < 0$. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \log^{-\varepsilon-\alpha}(n+2),$$

where $\varepsilon > 1$. Condition (4.1) holds because $\varepsilon > 1$. For every $r \in (0,1)$ the function $f_r(z) = f(rz)$ belongs to $H(\overline{\mathbb{D}})$ and, by Theorem 1.1 and its proof, the set $\{f_r : 0 < r < 1\}$ is bounded in $\mathfrak{B}^1_{\log^\alpha}$. On the other hand,

$$\overline{\mathcal{L}}(f_r)(0) = \sum_{k=0}^{\infty} \frac{r^k}{(k+1)\log^{\alpha+\varepsilon}(k+2)}.$$

Now choose $\varepsilon = 1 - \alpha > 1$ (because $\alpha < 0$) to get

$$\overline{\mathcal{L}}(f_r)(0) = \sum_{k=0}^{\infty} \frac{r^k}{(k+1)\log(k+2)}$$
$$\longrightarrow \infty \quad (r \uparrow 1).$$

This contradicts the fact that if a set $X \subset \mathfrak{B}^1_{\log^\alpha}$ is bounded and $\overline{\mathcal{L}}$ is bounded on $\mathfrak{B}^1_{\log^\alpha}$, then the set $\{\overline{\mathcal{L}}f(0) : f \in X\}$ is bounded because the functional $h \mapsto h(0)$ is continuous on $H(\mathbb{D})$. This completes the proof. $\square$

Proof of Theorem 2.1(a). Let $g \in \mathfrak{B}_{\log^\alpha}$. Then

$$M_2(r, g') \le C(1-r)^{-1}\log\frac{2}{1-r}.$$

It follows that

$$2^n \left(\sum_{k=2^n}^{2^{n+1}-1} |\hat{g}(k)|^2 \right)^{1/2} r^{2^{n+1}} \le C(1-r)^{-1}\log^\alpha \frac{2}{1-r}.$$

Taking $r = 1 - 2^{-n}$, $n \ge 1$, we get

$$\left(\sum_{k=2^n}^{2^{n+1}-1} |\hat{g}(k)|^2 \right)^{1/2} \le C\log^\alpha(n+1).$$

Hence,

$$2^{-n} \sum_{k=2^n}^{2^{n+1}-1} |\hat{g}(k)| \le \left(2^{-n} \sum_{k=2^n}^{2^{n+1}-1} |\hat{g}(k)|^2 \right)^{1/2}$$
$$\le 2^{-n/2}\log^\alpha(n+1).$$

This gives the result. $\square$

Proof of Theorem 2.1(b). In this case we use Hardy's inequality as in the proof of Theorem 1.1 to get

$$\|g\|_{\mathfrak{B}^1_{\log^\alpha}} \geq c \sum_{n=0}^{\infty} \frac{|\hat{g}(n)| \log^\alpha (n+2)}{n+1}.$$

This proves the result because $\alpha \geq 0$. $\square$

5. Proofs of Theorem 2.2 and 2.3

Define the operator $\mathcal{R} : H(\mathbb{D}) \mapsto H(\mathbb{D})$ by

$$\mathcal{R}f(z) = \sum_{n=0}^{\infty} (n+1)\hat{f}(n)z^n = \frac{d}{dz}(zf(z)).$$

By using Theorem 3.1 and the relation

(5.1) $C^{-1}2^n\|V_n * f\|_p \leq \|V_n * \mathcal{R}f\|_p \leq C2^n\|V_n * f\|_p \quad (n \geq 0)$

, one proves that the norm in $\mathfrak{B}_{\log^\alpha}$ is equivalent to

$$\sup_{z\in\mathbb{D}}(1 - |z|) \log^{-\alpha} \frac{2}{1 - |z|}|\mathcal{R}f(z)|.$$

Proof of Theorem 2.2(a). Let $\alpha > -1$ and $f \in \mathfrak{B}_{\log^\alpha}$. Then, by integration,

$$|f(z)| \leq \log^{\alpha+1} \frac{1}{1 - |z|}.$$

Since

$$\mathcal{R}\mathcal{C}f(z) = \frac{f(z)}{1 - z},$$

we see that

$$|\mathcal{R}\mathcal{C}f(z)| \leq C(1 - |z|)^{-1} \log^{\alpha+1} \frac{1}{1 - |z|}.$$

The result follows. $\square$

Proof of Theorem 2.2(b). The function $\varphi(x) = x \log \log(4/x)$ is normal because $\lim_{x\to 0} x\varphi'(x)/\varphi(x) = 1$. Hence, arguing as in the proof of Theorem 3.1 we conclude that $f \in \mathfrak{B}_{\log g}$ if and only if

$$\sup_{n\geq 0} \|V_n * f\|_\infty / \log(n+2) < \infty.$$

Then using (5.1) we find that $g \in \mathfrak{B}_{\log g}$ if and only if

$$|\mathcal{R}g(z)| \leq C(1 - |z|)^{-1} \log \log \frac{4}{1 - |z|}.$$

The rest of the proof is the same as in the case of (a). $\square$

Remark 5.1. In the case of the little spaces the proofs are similar and is therefore omitted.

For the proof of Theorem 2.3 we need the following lemma [18]:

Lemma 5.1. *If $f \in \ell^1_{-1}$, then $\mathcal{L}f$ is well defined by* (1.10) *and the inequality*

(5.2) $rM_1(r, (\mathcal{L}f)') \leq 2(1-r)^{-1} \int_r^1 M_1(s, f')\, ds, \quad 0 < r < 1,$

holds.

Before passing to the proof observe that $\mathfrak{B}^1_{\log^\alpha} \subset \mathfrak{B}^1$ and $\mathfrak{B}^1_{\log g} \subset \mathfrak{B}^1$, and, since $\mathfrak{B}_1 \subset H^1$, we see that in all cases of Theorem 2.3 the operator $\mathcal{L}$ is well defined.

Proof of Theorem 2.3(a). We have, by (5.2),

$$\int_{\mathbb{D}} |(\mathcal{L}f)'(z)| \log^{\alpha-1} \frac{2}{1-|z|} \, dA(z)$$

$$= 2 \int_0^1 M_1(r, (\mathcal{L}f)') \log^{\alpha-1} \frac{2}{1-r} r \, dr$$

$$\leq 4 \int_0^1 (1-r)^{-1} \log^{\alpha-1} \frac{2}{1-r} \, dr \int_r^1 M_1(s, f') \, ds$$

$$= 4 \int_0^1 M_1(s, f') \, ds \int_0^s (1-r)^{-1} \log^{\alpha-1} \frac{2}{1-r} \, dr$$

$$\leq C \int_0^1 M_1(s, f') \log^\alpha \frac{2}{1-s} \, ds.$$

A standard application of the maximum modulus principle shows that the inequality remains valid if we replace ds with $s \, ds$. This gives the result. $\square$

The proofs of Theorem 2.3, (b) and (c), are similar and we omit them.

6. Proof of Theorem 2.4

We consider a more general situation. Let $X \subset H(\mathbb{D})$ (with continuous inclusion) be a Banach space such that the functions $f_w(z) = f(wz)$, $|w| \leq 1$, belong to X whenever $f \in X$, and $\sup_{|w| \leq 1} \|f_w\|_X \leq \|f\|_X$. Such a space is said to be *homogeneous* (see [2]). A homogeneous space satisfies the condition

(6.1) $\|V_n * f\|_X \leq C\|f\|_X, \quad f \in X,$

where C is independent of n and f.

If in addition

(6.2) $\lim_{r \uparrow 1} \|f - f_r\|_X = 0, \quad f \in X,$

then the dual of X can be identified with the space, X', of those $g \in H(\mathbb{D})$ for which the limit (1.3) exists for all $f \in X$ (see [1, 2]). Also, the dual of a homogeneous space X satisfying (6.2) can be realized as the space of coefficient multipliers, $(X, A(\mathbb{D}))$, from X to $A(\mathbb{D})$; in this case we have $(X, A(\mathbb{D})) = (X, H^\infty) =: X^*$ (see [2]). The norm in X^* is introduced as

$$\|g\|_{X^*} = \sup \|f * g\|_\infty : f \in X, \ \|f\|_X \leq 1\},$$

and, if X is homogeneous and satisfies (6.2), it is equal to

$$\|g\|_{X'} = \sup\{|\langle f, g \rangle| : f \in X, \ \|f\|_X \leq 1\}.$$

There is another way to express $\langle f, r \rangle$, when $f \in X$, X satisfies (6.2), and $g \in X'$; namely, in this case, the function $f * g$ belongs to $A(\mathbb{D})$, and we have $\langle f, g \rangle = (f * g)(1)$ (see [13, 2]).

We fix a sequence $\lambda = \{\lambda_n\}_0^\infty$ of positive real numbers such that

(6.3) $0 < \inf_{n \geq 0} \frac{\lambda_{n+1}}{\lambda_n}, \quad \sup_{n \geq 0} \frac{\lambda_{n+1}}{\lambda_n} < \infty.$

It is clear that the spaces H^p $(0 < p \leq \infty)$, $A(\mathbb{D})$, $\mathfrak{B}_{\log^\alpha}$, $\mathfrak{b}_{\log^\alpha}$, and $\mathfrak{B}_{\log^\alpha}^1$ are homogeneous. Among them only H^∞ and $\mathfrak{B}_{\log^\alpha}$ do not satisfy condition (6.2).

Consider the following three spaces of sequences $\{f_n\}_0^\infty$, $f_n \in H(\mathbb{D})$:

(a) $\mathfrak{c}_0(\lambda, X) = \{\{f_n\} : \lim_{n\to\infty} \|f_n\|_X/\lambda_n = 0\}$;

(b) $\ell^\infty(\lambda, X) = \{\{f_n\} : \sup_{n\geq 0} \lambda_n \|V_n * f\|_X < \infty\}$;

(c) $\ell^1(\lambda, X) = \{\{f_n\} : \sum_{n=0}^\infty \|f_n\|_X/\lambda_n < \infty\}$.

We also define the spaces $v_0(\lambda, X)$, $V^\infty(\lambda, X)$, and $V^1(\lambda, X)$ (as subsets of $H(\mathbb{D})$) by replacing f_n with $V_n * f$ in (a), (b), and (c), respectively. The proof of the following lemma is rather easy, and is therefore left to the reader.

Lemma 6.1. *If X is a homogeneous space, then so are $v(\lambda, X)$, $V^\infty(\lambda, X)$, and $V^1(\lambda, X)$. The spaces $v_0(\lambda, X)$ and $V^1(\lambda, X)$ satisfy (6.2). The space $v_0(\lambda, X)$ is equal to the closure in $V^\infty(\lambda, X)$ of the sets of all polynomials.*

Theorem 2.4 will be deduced from Theorem 3.1 and the following.

Proposition 6.1. *If X is a homogeneous space satisfying (6.2), then the dual of $v_0(\lambda, X)$, resp. $V^1(\lambda, X)$, is isomorphic to $V^1(\lambda, X')$, resp. $V^\infty(\lambda, X')$, with respect to (1.3).*

In proving we use ideas from [13, 14, 7]. For the proof we need the following lemma.

Lemma 6.2. *The operator $T(\{f_n\}) = \sum_{n=0}^\infty V_n * f_n$ acts as a bounded operator from Y to Z, where Y is one of the spaces $\mathfrak{c}_0(\lambda, X)$, $\ell^\infty(\lambda, X)$, and $\ell^1(\lambda, X)$, while Z is $v_0(\lambda, X)$, $V^\infty(\lambda, X)$, and $V^1(\lambda, X)$, respectively.*

Proof. We have
$$V_n * V_j = 0 \quad \text{for } |j - n| \geq 2$$

and hence
$$V_n * T(\{f_j\}) = \sum_{j=n-1}^{n+1} V_n * V_j * f_j, \quad n \geq 0,$$

where, by definition, $w_j = f_j = 0$ for $j < 0$. It follows that

$$\|V_n * T(\{f_j\})\|_X \leq C \sum_{j=n-1}^{n+1} \|f_j\|_X,$$

where we have used (6.1). Now the proof is easily completed by using (6.3). $\square$

Lemma 6.3. *Let $g \in (v_0(\lambda, X))'$, resp. $g \in (V^1(\lambda, X))'$, and define the operator S on $\mathfrak{c}_0(\lambda, X)$, resp. $\ell^1(\lambda, X)$, by*

$$S(\{f_n\}) = T(\{f_n\}) * g = \sum_{k=0}^\infty f_k * V_k * g.$$

Then S maps $\mathfrak{c}_0(\lambda, X)$, resp. $\ell^1(\lambda, X)$, into H^∞ and $\|S\| \leq C\|g\|_{(v_0(\lambda, X))'}$, resp. $\|S\| \leq C\|g\|_{(V^1(\lambda, X))'}$.

Proof. By the preceding lemma, we have

$$\|S(\{f_n\})\|_\infty \le \|T(\{f_n\})\|_{v_0(\lambda,X)}\|g\|_{(v_0(\lambda,X))*}$$
$$\le C\|\{f_n\}\|_{\mathfrak{c}_0(\lambda,X)}\|g\|_{(v_0(\lambda,X))*}.$$

This proves the result in one case. In the other case the proof is the same. □

Proof of Proposition 6.1. Define the polynomials P_n $(n \ge 0)$ by

$$P_n = V_{n-1} + V_n + V_{n+1}.$$

Hence

$$V_n = \sum_{j=0}^{\infty} V_j * V_n = (V_{n-1} + V_n + V_{n+1}) * V_n = P_n * V_n.$$

Let $f \in v_0(\lambda, X)$ and $g \in V^1(\lambda, X')$. It is easily verified that, when $0 < r < 1$,

$$(f * g)(z) = \sum_{n=0}^{\infty} (f * V_n * g)(z)$$
$$= \sum_{n=0}^{\infty} (P_n * f * V_n * g)(z), \quad z \in \mathbb{D}.$$

the series being absolutely convergent. Since

$$\|P_n * f_r * V_n * g\|_\infty \le \|P_n * f\|_X \|V_n * g\|_{X^*},$$

we have

$$\|f * g\|_\infty \le \sum_{n=0}^{\infty} \|P_n * f\|_X \|V_n * g\|_{X^*}$$
$$\le C \sum_{n=0}^{\infty} \|P_n * f\|_X \|V_n * g\|_{X^*}$$
$$= C \sum_{n=0}^{\infty} \left(\|P_n * f\|_X / \lambda_n\right) \left(\lambda_n \|V_n * g\|_{X^*}\right)$$
$$\le C\|f\|_{v_0(\lambda,X)}\|g\|_{V^1(\lambda,X^*)}$$

This proves the inclusion $V^1(\lambda, X') \subset (v_0(\lambda, X))'$.

To prove the converse, let $g \in (v_0(\lambda, X))^*$. Let S denote the operator defined in Lemma 6.3. By Lemma 6.3, S acts as a bounded operator from $\mathfrak{c}_0(\lambda, X)$ into H^∞ and we have $\|S\| \le C\|g\|_{v_0(\lambda,X)^*}$. Now it suffices to prove that

$$\|S\| \ge (1/2)\|\{g_n\}\|_{\ell^1(\lambda,X^*)} = (1/2)\|g\|_{V^1(\lambda,X)}.$$

For each $n \ge 0$ choose $f_n \in X$ so that $\|f_n\|_X = 1$ and $\langle f_n, g_n \rangle$ is a real number such that $\langle f_n, g_n \rangle \ge (1/2)\|g_n\|_{X^*}$. If $\{a_n\}$ is a finite sequence of nonnegative real numbers, then

$$S(\{a_n f_n\}) = \sum_{n=0}^{\infty} a_n \langle f_n, g_n \rangle$$
$$\ge (1/2) \sum_{n=0}^{\infty} a_n \|g_n\|_{X^*}$$

$$= (1/2) \sum_{n=0}^{\infty} (a_n/\lambda_n)\lambda_n \|g_n\|_{X^*}.$$

Hence, by taking the supremum over all $\{a_n\}$ such that $0 \le a_n \le \lambda_n$, we get $S\{\lambda_n f_n\} \ge (1/2)\sum_{n=0}^{\infty} \lambda_n\|g_n\|_{X^*}$. Since $\|\{a_n f_n\}\|_{\mathfrak{c}_0(\lambda,X)} \le 1$, where $a_n = \lambda_n$ for $0 \le n \le N$ ($N \in \mathbb{N}$) and $a_n = 0$ for $n > N$ we see that $\|S\| \ge (1/2)\|g\|_{V^1(\lambda,X^*)}$, as desired. This completes the proof that $v_0(\lambda, X)' = V^1(\lambda, X')$. In a similar way one proves that $V^1(\lambda, X)' = V^{\infty}(\lambda, X')$, which is all what was to be proved. $\square$

Proof of Theorem 2.4. First we prove that $(\mathfrak{b}_{\log^\alpha})' = \mathfrak{B}^1_{\log^\alpha}$. By Theorem 3.1, we have $\mathfrak{b}_{\log^\alpha} = v_0(\lambda, A(\mathbb{D}))$, where $\lambda_n = (n+1)^\alpha$. Hence, by Proposition 6.1, the dual of $\mathfrak{b}_{\log^\alpha}$ is isomorphic to $V^1(\lambda, A(\mathbb{D})')$. In order to estimate $\|V_n*g\|_{A(\mathbb{D})'}$ first observe that $H^1 \subset A(\mathbb{D})'$ and moreover $\|V_n * g\|_{A(\mathbb{D})'} \le \|V_n * g\|_1$. On the other hand, let Φ be a bounded linear functional on $A(\mathbb{D})$, let Φ_0 be the Hahn/Banach extension of Φ to $hC(\mathbb{D})$, and choose $g \in A(\mathbb{D})^{\mathrm{a}}$ so that $\Phi(f) = \langle f, g \rangle$ for all $f \in A(\mathbb{D})$. By the Riesz representation theorem, we have

$$\Phi_0(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{-i\theta}) \, d\mu(e^{i\theta})$$

$$= \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} f(re^{-i\theta})g(re^{i\theta}) \, d\theta$$

$$= \lim_{r \to 1^-} \sum_{n=0}^{\infty} \hat{f}(n)\hat{g}(n)r^{2n},$$

and $\|\mu\| = \|\Phi\| = \|\Phi_0\|$. In particular, taking $f(w) = (1 - zw)^{-1}$, where $z \in \mathbb{D}$ is fixed, we get

$$\frac{1}{2\pi} \int_0^{2\pi} (1 - e^{-i\theta}z)^{-1} d\mu(e^{i\theta}) = g(z).$$

Hence

$$\mathcal{R}^1 g(z) = \frac{1}{2\pi} \int_0^{2\pi} (1 - e^{-i\theta}z)^{-2} \, d\mu(e^{i\theta}),$$

and hence, by integration,

$$M_1(r, \mathcal{R}^1 f) \le \|\mu\|(1 - r^2)^{-1} = \|g\|_{A(\mathbb{D})'}(1 - r^2)^{-1}.$$

Now we proceed as in the proof of Theorem 3.1 to conclude that $\|V_n * g\|_1 \le C\|V_n * g\|_{A(\mathbb{D})'}$. It follows that $g \in (\mathfrak{b}_{\log^\alpha})'$ if and only if $g \in V^1(\lambda, H^1)$, i.e., by Theorem 3.1, $g \in \mathfrak{B}^1_{\log^\alpha}$.

In proving that $(\mathfrak{B}^1_{\log^\alpha})'$ is isomorphic to $\mathfrak{B}_{\log^\alpha}$ we use the inclusions $H^{\infty} \subset (H^1)' \subset \mathfrak{B}$, and then proceed as above. $\square$

Remark 6.1. The above proof of Theorem 2.4 certainly is not the simplest one. However, it can be applied to prove some general duality and multipliers theorems (see [13, 14, 7]. For instance, the dual of $\mathfrak{b}_{\log g}$ is isomorphic to $\mathfrak{B}^1_{\log g}$, and the dual of $\mathfrak{B}^1_{\log g}$ is isomorphic to $\mathfrak{B}_{\log g}$.

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